Learning the following before you start doing problems is necessary.

1. Note that $\sin^{-1} x \neq \frac{1}{\sin x}$ and $(\sin^{-1} x)^2 \neq \sin^{-2} x$. Also $\sin^{-1} x \neq (\sin x)^{-1}$.

2. Principal value branch of inverse trigonometric functions

Function	*Domain*	*Range* (*Principal value branch*)
(*i*) $y = \sin^{-1} x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
(*ii*) $y = \cos^{-1} x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
(*iii*) $y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
(*iv*) $y = \text{cosec}^{-1} x$	$x \geq 1$ or $x \leq -1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$
(*v*) $y = \sec^{-1} x$	$x \geq 1$ or $x \leq -1$	$0 \leq y \leq \pi, y \neq \frac{\pi}{2}$
(*vi*) $y = \cot^{-1} x$	$-\infty < x < \infty$	$0 < y < \pi$

3. ***FORMULAE TO BE PRACTICED***

(*i*)
- $\sin^{-1}(\sin x) = x, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$;
- $\sin(\sin^{-1} x) = x, x \in [-1, 1]$
- $\cos^{-1}(\cos x) = x$; $\cos(\cos^{-1} x) = x$
- $\tan^{-1}(\tan x) = x$; $\tan(\tan^{-1} x) = x$
- $\text{cosec}^{-1}(\text{cosec}\, x) = x$; $\text{cosec}(\text{cosec}^{-1} x) = x$
- $\sec^{-1}(\sec x) = x$; $\sec(\sec^{-1} x) = x$
- $\cot^{-1}(\cot x) = x$; $\cot(\cot^{-1} x) = x$

(*ii*)
- $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1} x, x \geq 1$ Or $x \leq -1$;
- $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x, x \geq 1$ or $x \leq -1$

- $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x, x > 0$; • $\operatorname{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1} x$
- $\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1} x$; • $\cot^{-1}\left(\frac{1}{x}\right) = \tan^{-1} x$

(*iii*)
- $\sin^{-1}(-x) = -\sin^{-1} x$; • $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$
- $\cos^{-1}(-x) = \pi - \cos^{-1} x$; • $\sec^{-1}(-x) = \pi - \sec^{-1} x$
- $\tan^{-1}(-x) = -\tan^{-1} x$; • $\cot^{-1}(-x) = \pi - \cot^{-1} x$

(*iv*)
- $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$; • $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}$
- $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

(*v*)
- $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$, if $xy < 1$
- $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy}$, $xy > -1$

(*vi*) $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2\tan^{-1} x.$

(*i*) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

(*ii*) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$

(*iii*) $\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$

(*iv*) $\cos^{-1} x - \cos^{-1} y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$

PRACTICE – 1 – FINDING PRINCIPAL VALUE BRANCH OF INVERSE TRIGONOMETRIC FUNCTION

1. Find the Principal Value Branch of the following :

(i) $\sin^{-1}(-1)$ (ii) $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

Solution:

(i) Let $\sin^{-1}(-1) = \theta,\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \sin\theta = -1 = \sin\left(\frac{-\pi}{2}\right)$

$\Rightarrow \theta = \frac{-\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \sin^{-1}(-1) = \frac{-\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow$ P. value of $\sin^{-1}(-1) = \frac{-\pi}{2}$.

(ii) Let $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \theta,\ \theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \tan\theta = \frac{-1}{\sqrt{3}} = \tan\left(\frac{-\pi}{6}\right) \Rightarrow \theta = \frac{-\pi}{6}$

$\Rightarrow \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \frac{-\pi}{6} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \Rightarrow$ P. value of $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \frac{-\pi}{6}$

PRACTICE YOURSELF – 1

Find the Principal Value Branch of the following functions

(i) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (ii) $\cot^{-1}(\sqrt{3})$

2. *Find the principal value of :*

(i) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ (ii) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

Sol. (i) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) \neq \frac{2\pi}{3}$ because $\frac{2\pi}{3}$ does not lie between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$

$$\therefore \sin^{-1}\left(\sin\frac{2\pi}{3}\right)=\sin^{-1}\left[\sin\left(\pi-\frac{\pi}{3}\right)\right]=\sin^{-1}\sin\frac{\pi}{3}=\frac{\pi}{3}.$$

(ii) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) \neq \frac{7\pi}{6}$ because $\frac{7\pi}{6}$ does not lie between 0 and π.

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right)=\cos^{-1}\left[\cos\left(2\pi-\frac{5\pi}{6}\right)\right]=\cos^{-1}\left[\cos\frac{5\pi}{6}\right]=\frac{5\pi}{6}$$

3. *Find the value of* $\tan^{-1} 1+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

Sol.

Let $\tan^{-1} 1 = \theta \Rightarrow \tan\theta = 1 = \tan\frac{\pi}{4} \quad \therefore \quad \theta = \frac{\pi}{4}$

$\therefore \quad \tan^{-1} 1 = \frac{\pi}{4}$ ...(1)

Let $\cos^{-1}\left(-\frac{1}{2}\right) = \theta,\ \theta \in (0, \pi)$

$\Rightarrow \quad \cos\theta = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi-\frac{\pi}{3}\right) = \cos\frac{2\pi}{3} \quad \Rightarrow \quad \theta = \frac{2\pi}{3}$

$\Rightarrow \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ ...(2)

Let $\sin^{-1}\left(-\frac{1}{2}\right) = \theta,\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad \Rightarrow \quad \sin\theta = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin\left(-\frac{\pi}{6}\right)$

$\therefore \quad \theta = -\frac{\pi}{6}$

$\therefore \quad \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$\therefore \quad$ given result $= \frac{\pi}{4}+\frac{2\pi}{3}-\frac{\pi}{6} = \frac{3\pi+8\pi-2\pi}{12} = \frac{9\pi}{12} = \frac{3\pi}{4}$

4. *Find the value of* $\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$

Sol.

Let $\cos^{-1}\frac{1}{2}=\theta,\ \theta\in(0,\pi)$

$\Rightarrow \quad \frac{1}{2}=\cos\theta \Rightarrow \cos\theta=\cos\frac{\pi}{3}\Rightarrow\theta=\frac{\pi}{3}$

$\therefore \quad \cos^{-1}\frac{1}{2}=\frac{\pi}{3}$

Let $\sin^{-1}\frac{1}{2}=\theta,\ \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \Rightarrow \sin\theta=\frac{1}{2}=\sin\frac{\pi}{6} \Rightarrow \theta=\frac{\pi}{6}$

$\therefore \quad \sin^{-1}\frac{1}{2}=\frac{\pi}{6}\Rightarrow 2\sin^{-1}\frac{1}{2}=2\left(\frac{\pi}{6}\right)=\frac{\pi}{3}$

$\therefore$ given expression $=\frac{\pi}{3}+\frac{\pi}{3}=\frac{2\pi}{3}$

Using Formulae evaluating Inverse Trigonometric Functions

1. *Write* $\sin^{-1}\left(2x\sqrt{1-x^2}\right),\ -\frac{1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}$ *in the simplest form.*

Sol. Put $x=\sin\theta$

$\therefore\ 2x\sqrt{1-x^2}=2\sin\theta.\sqrt{1-\sin^2\theta}=2\sin\theta\cos\theta=\sin 2\theta$

$\therefore\ \sin^{-1}\left(2x\sqrt{1-x^2}\right)=\sin^{-1}(\sin 2\theta)=2\theta=2\sin^{-1}x.$

2. *Show that* $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{2}{11} = \tan^{-1}\frac{3}{4}$.

Solution

Formula used

$$=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{2}{11} = \tan^{-1}\frac{\frac{1}{2}+\frac{2}{11}}{1-\frac{1}{2}.\frac{2}{11}} \quad \left(\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right)$$

$$= \tan^{-1}\frac{\frac{15}{22}}{\frac{10}{11}} = \tan^{-1}\frac{15}{20} = \tan^{-1}\frac{3}{4} = \text{R.H.S.}$$

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3. *Express* $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right),\ -\frac{\pi}{2}<x<\frac{\pi}{2}$ *in the simplest form.*

Solution

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \tan^{-1}\left(\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2.\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$$= \tan^{-1}\frac{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2} = \tan^{-1}\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}$$

$$= \tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right) = \tan^{-1}\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) = \frac{\pi}{4}+\frac{x}{2}$$

4. *Prove that* $\tan^{-1} x + \tan^{-1}\dfrac{2x}{1-x^2} = \tan^{-1}\dfrac{3x-x^3}{1-3x^2}$, $x^2 < \dfrac{1}{3}$.

Solution:

Let $\tan^{-1} x = \theta$, $\therefore\ x = \tan\theta$

$$\text{L.H.S.} = \tan^{-1} x + \tan^{-1}\frac{2x}{1-x^2} = \theta + \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$= \theta + \tan^{-1}(\tan 2\theta) = \theta + 2\theta = 3\theta \qquad \ldots(1)$$

$$\text{R.H.S.} = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 3\theta) = 3\theta \qquad \ldots(2)$$

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5. *Find the value of each of the following :*

(i) $\cot(\tan^{-1} a + \cot^{-1} a)$ *(ii)* $\sin(\sin^{-1} x + \cos^{-1} x)$

(iii) $\cos(\sec^{-1} x + \mathrm{cosec}^{-1} x)$

Solution:

(*i*) $\cot(\tan^{-1} a + \cot^{-1} a) = \cot\left(\dfrac{\pi}{2}\right) = 0$

(*ii*) $\sin(\sin^{-1} x + \cos^{-1} x) = \sin\left(\dfrac{\pi}{2}\right) = 1$

(*iii*) $\cos(\sec^{-1} x + \mathrm{cosec}^{-1} x) = \cos\left(\dfrac{\pi}{2}\right) = 0$

6. Prove that: $\tan^{-1}\dfrac{2}{11} + \tan^{-1}\dfrac{7}{24} = \tan^{-1}\dfrac{1}{2}$

Solution:

$$\text{L.H.S.} = \tan^{-1}\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\cdot\frac{7}{24}} = \tan^{-1}\left(\frac{\frac{48+77}{264}}{\frac{264-14}{264}}\right)$$

$$= \tan^{-1}\left(\frac{125}{250}\right) = \tan^{-1}\frac{1}{2} = \text{R.H.S.}$$

$\therefore$ L.H.S. = R.H.S.

CHAPTER AT A GLANCE

1 Mark

1) Find the value of $\cos\left\{\text{Cos}^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\Pi}{6}\right\}$ Ans: -1

2) Find the value of $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3$. Ans: Π

3) Solve for x : $\text{Sin}\left[\text{Sin}^{-1}\frac{1}{5}+\text{Cos}^{-1}x\right]=1$ Ans: $\frac{1}{5}$

4) Write the simplest form : $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$ Ans: $\frac{\Pi}{4}+\frac{x}{2}$

5) Considering the principal solutions, find the number of solutions of

$\tan^{-1}2x+\tan^{-1}3x=\frac{\Pi}{4}$ Ans: 2

6) Find the principal value of $\text{Sin}^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\text{Cos}^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ Ans: $\frac{\Pi}{2}$

7) Find the value of x if $\text{Cosec}^{-1}x+2\text{Cot}^{-1}7+\text{Cos}^{-1}\frac{3}{4}$ Ans: $x=\text{Cosec}^{-1}\frac{125}{117}$

8) If $\text{Cos}^{-1}x = \tan^{-1}x$, show that $\text{Sin}(\cos^{-1}x) = x^2$

9) If x > 0 and $\text{Sin}^{-1}\left(\frac{5}{x}\right)+\text{Sin}^{-1}\left(\frac{12}{x}\right)=\frac{\Pi}{2}$, then find the value of x. Ans: x = 13

10) Prove that $\text{Cos}\left\{2\text{Cot}^{-1}\sqrt{\frac{1-x}{1+x}}\right\}+x=0$

4 Marks / 6 Marks

11) Prove that $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=\frac{\Pi}{4}$

12) If $x, y, z \,\varepsilon\, [-1,1]$ such that $\text{Sin}^{-1}x+\text{Sin}^{-1}y+\text{Sin}^{-1}z=\frac{3\Pi}{2}$, find the value of

$x^{2006}+y^{2007}+z^{2008}-\frac{9}{x^{2006}+y^{2007}+z^{2008}}$ Ans: zero ; x=1,y=1,z=1

13) If $\text{Cos}^{-1}\frac{x}{a}+\text{Cos}^{-1}\frac{y}{b}=\alpha$, prove that $\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha$

14) If $\text{Sin}^{-1}x+\text{Sin}^{-1}y+\text{Sin}^{-1}z=\Pi$, prove that $x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}=2xyz$

15) Prove that : $\text{Sin}2\left[\cot^{-1}\left\{\cos(\tan^{-1}x)\right\}\right]=\sqrt{\frac{x^2+1}{x^2+2}}$

16) In any Triangle ABC, if $A = \tan^{-1}2$ and $B = \tan^{-1}3$, prove that $C=\frac{\Pi}{4}$

RAPID FIRE REVISION ON INVERSE TRIGONOMETRIC FUNCTIONS

Q. 1.

$$\text{Prove that } \cos^{-1} x = 2\sin^{-1}\sqrt{\frac{1-x}{2}}.$$

Q. 2.

$$\text{Prove that } \tan^{-1} x = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x \in (0, 1).$$

Q. 3.

$$\text{Prove that } \tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2},$$

Q. 4.

$$\text{Prove that } \cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65},$$

Q. 5. $\text{Evaluate } \sin\ (\cot^{-1} x)$

Q. 6. Find the Value of

i. $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$

ii. $\tan^{-1}\left(\sin\frac{3\pi}{4}\right)$ -

Q. 7. Write the following in simplest form :

i. $\tan^{-1}\frac{\sqrt{1+x^2-1}}{x}, x \neq 0$

ii. $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$

Q. 8.

$$\text{Simplify} : \tan^{-1}\left[\frac{a\cos x = b\sin x}{b\cos x + a\sin x}\right], \text{if } \frac{a}{b}\tan x > -1.$$

Q. 9. Find the principal value of :

i. $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

ii. $\cos^{-1}\left(-\frac{1}{2}\right)$

Chapter - 2: Answers

5. $\sqrt{1-x^2}$

6. **(i)** $\frac{2\pi}{5}$ **(ii)** $-\frac{\pi}{4}$

7. **(i)** $\frac{1}{2}\tan^{-1}x$ **(ii)** $\frac{\pi}{4}-x$

8. $\tan^{-1}\frac{a}{b}-x$

9. **(i)** $\frac{\pi}{4}$ **(ii)** $\frac{2\pi}{3}$