APPLICATIONS OF DERIVATIVE

In this topic, we would be learning the following topics:

1. Rate of Chage of quantities
2. Increasing & Decreasing Functions
3. Rolle’s Theorem
4. LMV Theorem
5. Maxima and Minima

RATE OF CHANGE OF QUANTITIES:

If a quantity y varies with respect to another quantity 'x' satisfying some rule $y = f(x)$, in other words if y is a function x, then $\frac{dy}{dx}$ (or $f'(x)$) represents the rate of change of y w.r.t. x

For $x = x_0$,

$\frac{dy}{dx}$ at x_0 is called the rate of change of y with respect to x at x_0.

If y is a function of t and x is a function of t

That is if $x = f(t)$ and $y = g(t)$

We know that $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ provided $\frac{dx}{dt} \neq 0$

The rate of change of y with respect to x in this case, can be found by finding out the rate of change of y with respect to t and that of x with respect to

1. **A balloon which always remain spherical is being inflated by pumping in 900 cubic centimeters of gas per second. Find the rate of change at which the radius of the balloon increasing when the radius is 15 cm**

Solution :

Let r and V be the variables for radius and volume of the balloon respectively.

Given is $\frac{dV}{dt} = 900$ cc/sec

We have to find

$\frac{dr}{dt}$ when r = 15cm

For a sphere

$V = \frac{4}{3}\pi r^3$

Differentiating both sides with respect to "t", we have

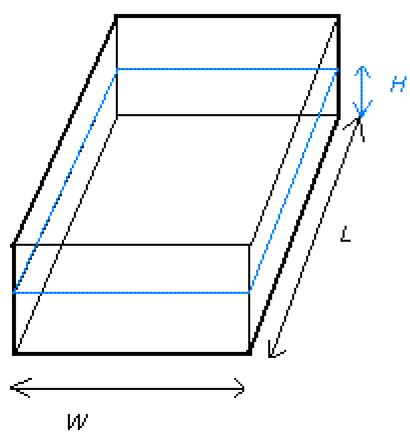
$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{\frac{dv}{dt}}{4\pi r^2} = \frac{900}{4x\pi x15x15}$$

$$= \frac{1}{\pi} \text{cm/sec.}$$

Problem 2:

A rectangular water tank (see figure below) is being filled at the constant rate of 20 liters / second. The base of the tank has dimensions w = 1 meter and L = 2 meters. What is the rate of change of the height of water in the tank?(express the answer in cm / sec).



Solution:

The volume V of water in the tank is given by.

V = L x B x H

We know the rate of change of the volume dV/dt = 20 liter /sec. We need to find the rate of change of the height H of water dH/dt. V and H are functions of time. We can differentiate both side of the above formula to obtain

$$\frac{dV}{dt} = L \times B \times \frac{dH}{dt}$$

We now find a formula for $\frac{dH}{dt}$ as follows.

$$\frac{dH}{dt} = L \times B \times \frac{dH}{dt}$$

We need to convert liters into cubic cm and meters into cm as follows

1 liter = 1 cubic decimeter
= 1000 cubic centimeters
= 1000 cm^3

and 1 meter = 100 centimeter.

We now evaluate the rate of change of the height H of water.

$$\frac{dH}{dt} = L \times B \times \frac{dH}{dt}$$

$$= \frac{(20 \times 1000\ cm\,3\ /\ sec)}{(100\ cm \times 200\ cm)}$$

$$= 1\ cm\ /\ sec.$$

1. A function is *increasing* on an interval if for any x_1 and x_2 in the interval then
   $x_1 < x_2$ implies $f(x_1) < f(x_2)$

2. A function is *decreasing* on an interval if for any x_1 and x_2 in the interval then
   $x_1 < x_2$ implies $f(x_1) > f(x_2)$

Alternate Definition for Increasing & Decreasing Functions

Let f be a differentiable function on the interval (a, b) then

1. If $f'(x) < 0$ for x in (a, b), then f is decreasing there.
2. If $f'(x) > 0$ for x in (a, b), then f is increasing there.
3. If $f'(x) = 0$ for x in (a, b), then f is constant

Problem 1:

Determine the values of x where the function $f(x) = 2x^3 + 3x^2 - 12x + 7$

Solution

We first take the derivative

$$f'(x) = 6x^2 + 6x - 12$$

To determine where the derivative is positive and where it is negative, find the roots. Factor to get;

$$6(x^2 + x - 2) = 6(x - 1).(x + 2)$$

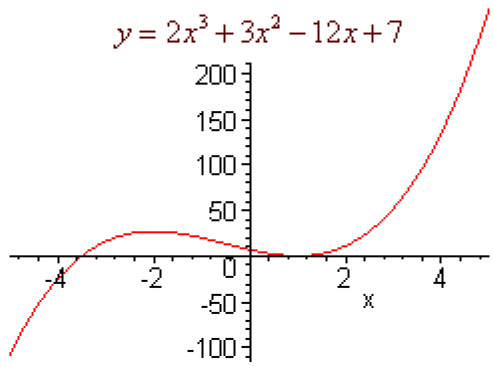
Hence the change in sign can occur when

$x = 1$ and $x = -2$

Now create some test values

x	f '(x)
-3	24
0	-12
2	24

The derivative is positive outside of [-2, 1] and is negative inside of [-2, 1]. We can conclude that f is increasing outside of [-2, 1] and decreasing inside of [-2, 1]. The graph is shown below.



We saw that the values of x such that the derivative is 0 were of special interest. Other points where there could be a change from increasing to decreasing is where the derivative is undefined.

We call c a *critical number* if either f '(c) = 0 or f '(c) is undefined.

Problem No. 2

Determine where the function below is increasing and where it is decreasing.

$$f(x) = \frac{2}{x - 1} + 18x$$

$$f(x) = \frac{2}{x - 1} + 18x$$

Solution

Since f(x) is not continuous at x = 1, it is also not differentiable there. Hence x=1 is a critical point. To find other critical points, we take a derivative. It is helpful to use negative exponents instead of fractions here.

$$f'(x) = \frac{d\left[2(x-1)^{-1} + 18x\right]}{dx}$$

$$= -2(x-1)^{-2} + 18$$

$f'(x) = 0$ *to obtain Critical points*

$18 = 2(x - 1)^{-2}$ Divide by 2 and multiplying by $(x - 1)^2$

$9(x - 1)^2 = 1$ Take the square root of both sides

$3(x - 1) = 1$ or $3(x - 1) = -1$

$x = 4/3$ or $x = 2/3$

This gives us three critical points

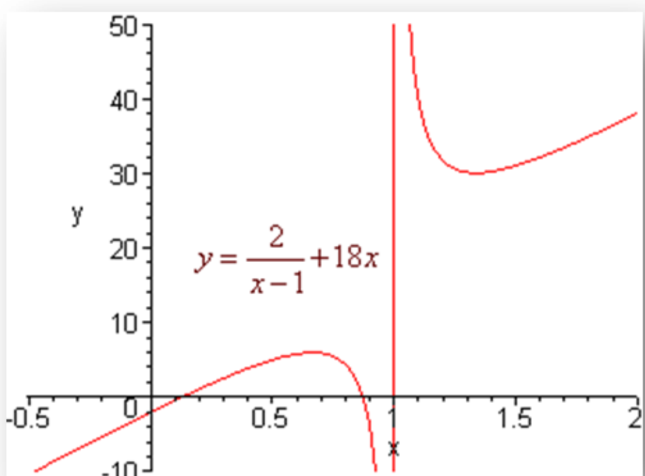
$x = 2/3$ $x = 1$ and $x = 4/3$

Now construct a table and determine positive intervals and negative intervals

$-2(x - 1)^{-2} + 18$

x	**f '(x)**
0	Positive
.9	Negative
1.1	Negative
2	Positive

We can conclude that f is increasing for values of x less than 2/3 and values of x greater then 4/3. f is decreasing for values between 2/3 and 4/3 excluding x = 1. Observe the graph to understand **Increasing and decreasing behaviour** of the function.



TEST YOURSELF

Determine the intervals in which the functions given beneath are increasing or decreasing
$f(x) = 2x^3 - 9x^2 - 108x + 2$
$f(x) = x^4 - x^3 + 1$
$f(x) = \|x^2 - 9\|$
$f(x) = \ln(x^2 + 1)$

REFERENCES FOR YOU ON THE ABOVE TEST

Solution No. 1:

We are given the function

$$f(x) = 2x^3 - 9x^2 - 108x + 2$$

First, we find the derivative:

$$f'(x) = 6x^2 - 18x - 108$$

We set the derivative equal to 0 and solve:

$$6x^2 - 18x - 108 = 0$$

$$6(x^2 - 3x - 18) = 0$$

$$(x - 6)(x + 3) = 0$$

$$x = -3$$

$$x = 6$$

Since the domain of **f** is the same as the domain of **f'**, **-3** and **6** are the only critical numbers of **f**.

Testing:

x < -3	**f'(-10) = 672**	**f** is increasing
-3 < x < 6	**f(0) = -108**	**f** is decreasing
x >6	**f(10) = 312**	**f** is increasing

Solution: 2

We are given the function

$$f(x) = x^4 - x^3 + 1$$

First, we find the derivative:

$$f'(x) = 4x^3 - 3x^2$$

We set the derivative equal to 0 and solve:

$$4x^3 - 3x^2 = 0$$

$$x^2(4x - 3) = 0$$

$$x = \frac{3}{4}$$

$$x = 0$$

Since the domain of **f** is the same as the domain of **f'**, **0** and **0.75** are the only critical numbers of **f**.
Testing:

x < 0	**f'(-1) = -7**	**f** is decreasing
0 < x < 0.75	**f(0.5) = -0.25**	**f** is decreasing
x >0.75	**f(1) = 1**	**f** is increasing

Solution No.3

We are given the function

$$f(x) = |x^2 - 9|$$

First, we find the derivative:

$$f'(x) = 2x \ (when \ x < -3)$$

$$f'(x) = -2x \ (when \ -3 < x < 3)$$

$$f'(x) = 2x \ (when \ x > 3)$$

We set the derivative equal to 0 and solve:

$$-2x = 0$$

$$x = 0$$

Since **f'** is not defined at **- 3** and **3** which are in the domain of **f**, **-3**, **0** and **3** are the critical numbers of **f**.

Testing:

x < -3	**f'(-10) = -20**	**f** is decreasing
-3 < x < 0	**f(-1) = 2**	**f** is increasing
0 < x < 3	**f(1) = -2**	**f** is decreasing
x > 3	**f(10) = 20**	**f** is increasing

Rolle's Theorem

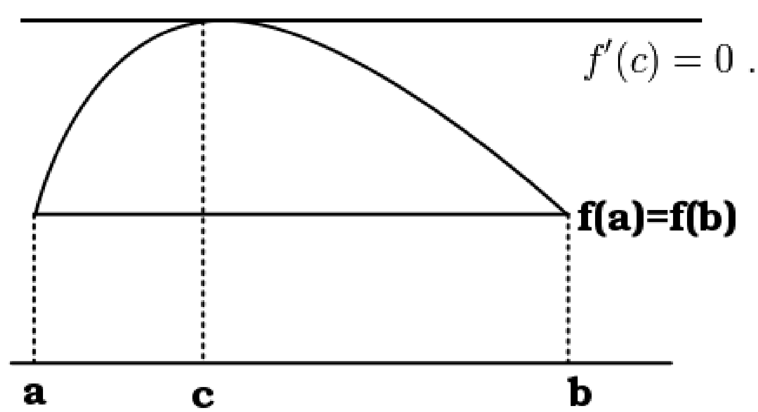
Let $f(x)$ be a function. Rolle's Theorem is said to be applicable

i. is continuous on the closed interval $[a, b]$

ii. is differentiable on the open interval (a, b); and

iii. $f(a) = f(b)$

Then there exists at least one number c, in the interval (a, b) such that $f'(c) = 0$.



implies the slope of the tangent drawn at c is zero. So tangent is parallel to x-axis at

$f'(c) = 0$ point "c"

Problem No.1

Does Rolle's Theorem apply $f(x) = \sqrt{9 - x^2}$ **to define on [-3, 3]? If so, find the number that exists in the interval [-3, 3]**

Solution:
The function

i. is function is continuous on the interval [-3, 3],

ii. is differentiable on (-3, 3). It is not differentiable at $x = -3$ and $x = +3$, but Rolle's Theorem does not require the function to be differentiable at the endpoints.

iii. Also $f(3) = f(-3) = 0$. Therefore, Rolle's Theorem does apply

It guaranties the existence of a number c between -3 and 3 such that $f'(c) = 0$.

PRACTICE YOURSELF

1. $f(x) = x^2 - 5x + 6$ in the interval [2, 3].
2. $f(x) = x^3 - 7x^2 + 16x - 12$ in the interval [2, 3].
3. $f(x) = x^{2/3}$ on [– 1, 1].
4. $f(x) = x^3 - 9x^2 + 26x - 24$ in the interval [2, 4].
5. $f(x) = \cos x + \sin x$ on $[0, 2\pi]$.

Answers

1. $c = \frac{5}{2}$
2. $c = \frac{8}{3}$
3. Not applicable
4. Not applicable
5. $c = \frac{\pi}{4}, \frac{5\pi}{4}$

LAGRANGE'S MEAN VALUE THEOREM (LMV)

Problem on LMV
For each of the following functions, verify Lagrange's Mean Value Theorem

1.	$f(x) = x^2 - 5x + 7, \ -1 \le x \le 3$
2.	$f(x) = x^3 - 6x^2 + 9x + 2, \ 0 \le x \le 4$

3.	$f(x) = sin2x + cosx\ ;\ [0, \pi]$
4.	$f(x) = x^4 - 16x^2 + 2, -1 \le x \le 3$

Solutions to above Problems:

To apply the Mean Value Theorem to the function

$$f(x) = x^2 - 5x + 7,\ -1 \le x \le 3$$

We first calculate the quotient

$$\frac{f(3) - f(-1)}{3 - (-1)} = \frac{1 - 13}{4} = -3$$

Next, we take the derivative

$$f'(x) = 2x - 5$$

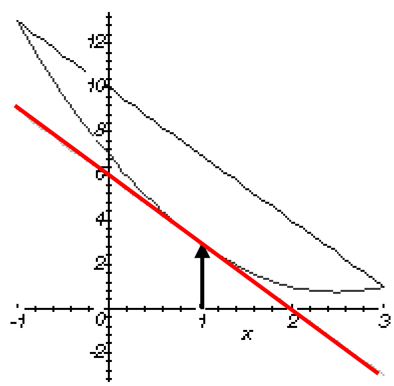
And equate it to the result of the calculation above:

$$2x - 5 = -3$$

We solve this equation to get

x = 1.

The following figure explains the above solution geometrically. The LMV is justified at the point x = 1



Solution No. 2

To apply the Mean Value Theorem to the function

$$f(x) = x^3 - 6x^2 + 9x + 2,\ 0 \le x \le 4$$

We first calculate the quotient

$$\frac{f(4) - f(0)}{4 - 0} = \frac{6 - 2}{4} = 1$$

Next, we take the derivative

$$f'(x) = 3x^2 - 12x + 9$$

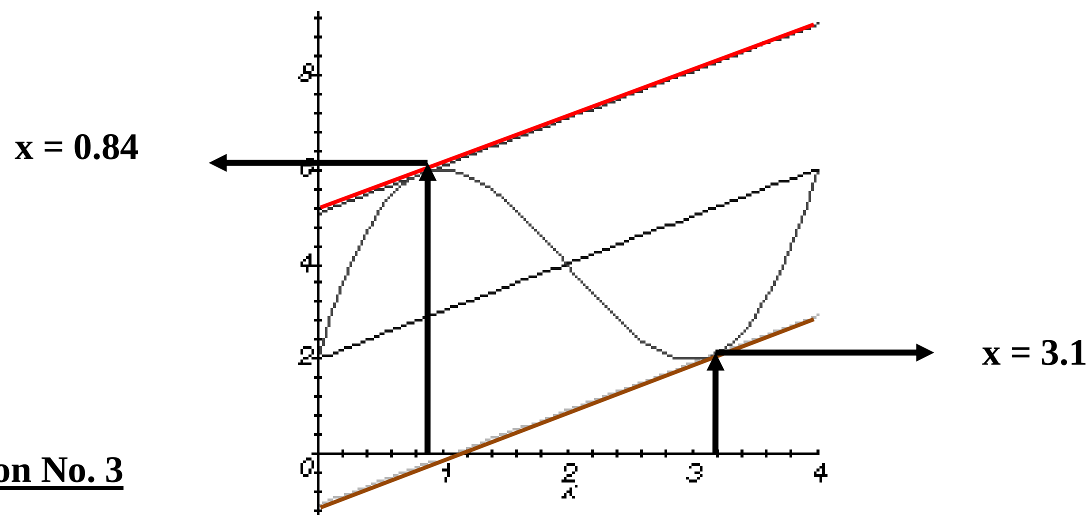
And equate it to the result of the calculation above:

$$3x^2 - 12x + 9 = 1$$

We solve this equation to get two answers:

x = 3.1 and **x = 0.84**

<u>The following figure clearly shows two tangents to the curve f(x) justifying LMV theorem</u>



<u>Solution No. 3</u>

To apply the Mean Value Theorem to the function

$$f(x) = \sin(2x) + \cos(x),\ 0 \le x \le \pi$$

We first calculate the quotient

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{-1-1}{\pi} = -\frac{2}{\pi}$$

Next, we take the derivative

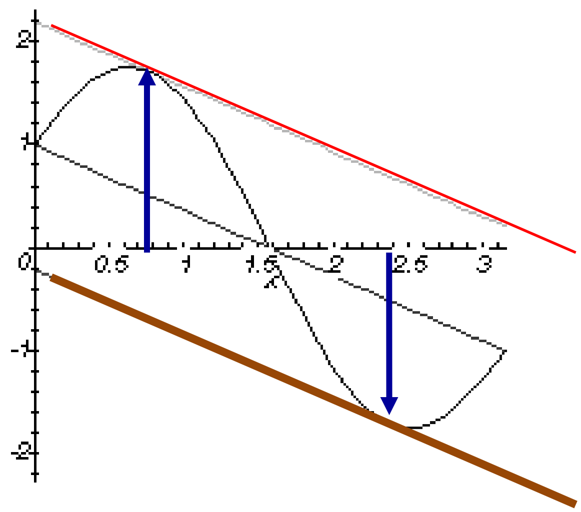
$$f'(x) = 2\cos(2x) - \sin(x)$$

And equate it to the result of the calculation above:

$$2\cos(2x) - \sin(x) = -\frac{2}{\pi}$$

We solve this equation to get two answers:

x = 0.77 and **x = 2.37**



Solution No.4

Workout the last problem to get the answer

x = 0.38 and x= **2.6**

Solution No. 4

We are given the function

$$f(x) = \ln(x^2 + 1)$$

First, we find the derivative:

$$f'(x) = \frac{2x}{x^2 + 1}$$

We set the derivative equal to 0 and solve:

$$\frac{2x}{x^2 + 1} = 0$$

$$x = 0$$

Since the domain of **f** is the same as the domain of **f'**, **0** is the only critical number of **f**.

Testing:

x < **0**	**f'(-1) = -1**	**f** is decreasing
x > **0**	**f'(1) = 1**	**f** is increasing

MAXIMA AND MINIMA

2^{ND} Derivative Test of Maxima & Minima

The diagram below shows part of a function y = f(x). The point A is a local maximum and the point B is a local minimum. At each of these points the tangent to the curve is parallel to the x-axis so the derivative of the function is zero. Both of these points are therefore stationary points of the function. The term local is used since these points are the maximum and minimum in this particular region. There may be others outside this region.

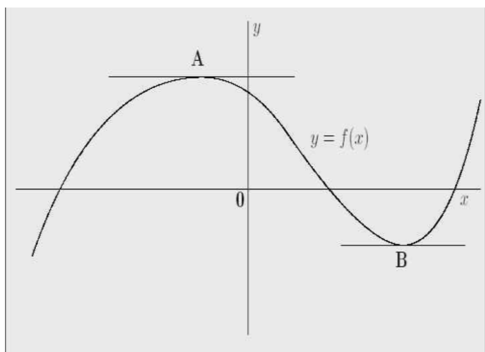


Diagram 1

Problem No.1

Find two nonnegative numbers whose sum is 9 and so that the product of one number and the square of the other number is a maximum.

Let variables x and y represent two nonnegative numbers. The sum of the two numbers is given to be

$$9 = x + y$$

so that

$$y = 9 - x$$

We wish to MAXIMIZE the PRODUCT

$$P = x\,y^2$$

However, before we differentiate the right-hand side, we will write it as a function of x only. Substitute for y getting $P = x\,y^2$

$= x(9-x)^2$.

Now differentiate this equation using the product rule and chain rule, getting

$P' = x(2)(9-x)(-1) + (1)(9-x)^2$

$= (9-x)[-2x + (9-x)]$

$= (9-x)[9-3x]$

$= (9-x)(3)[3-x]$

$= 0$; for $x=9$ or $x=3$.

Note that since both x and y are nonnegative numbers and their sum is 9, it follows that $0 \le x \le 9$

.

If $x=3$ and $y=6$, then $P= 108$ which is maximum

<u>Problemson Maxima and Minima for practice</u>

1. An open rectangular box with square base is to be made from 48 ft.^{2} of material. What dimensions will result in a box with the largest possible volume? [Ans. $V = 32$ ft.^{3}]

2. Construct a window in the shape of a semi-circle over a rectangle. If the distance around the outside of the window is 12 feet, what dimensions will result in the rectangle having largest possible area? [$A \approx 6.99$ ft.^{2} is the largest possible area of the rectangle.]

3. Show that the height of the right circular cylinder of maximum volume that can be inscribed in a given right circular cone of height h is $\frac{h}{3}$.
4. Prove that the function $f(x) = x^3 + x^2 + x + 1$ does not have a maxima or minima.
5. Find a point on the parabola $y^2 = 2x$ which is closest to the point (1, 4).
6. A window is in the form of a rectangle surmounted by a semicircular opening. If the perimeter of window is 20 m, find the dimensions of the window so that the maximum possible light is admitted through the whole opening.
7. Find the largest possible area of a right angled triangle whose hypotenuse is 5 cm long.

APPLICATION OF DERIVATIVES

1) The slope of the tangent to the curve represented by $x = t^2+3t-8$ and $y = 2t^2-2t-5$ at the point M(2,-1) is

(a) $7/6$ (b) $2/3$ (c) $3/2$ (d) $6/7$ Ans (d)

2) The function $f(x) = 2 \log (x-2) - x^2+4x+1$ increases in the interval.

(a) (1,2) (b) (2,3) (c) $\left(5/2, 3\right)$ (d) (2,4) Ans: (b) and (c)

3) The function $y = \tan^{-1}x - x$ decreases in the interval of

(a) $(1, \infty)$ (b) $(-1, \infty)$ (c) $(-\infty, \infty)$ (d) $(0, \infty)$ Ans: all

4) The value of a for which the function $f(x) = a\text{Sin}x + \frac{1}{3}\text{Sin}3x$ has an extreme at $x = \Pi/3$ is

(a) 1 (b) -1 (c) (0) (d) 2 Ans: d

5) The co-ordinates of the point p(x,y) in the first quadrant on the ellipse $\frac{x^2}{8}+\frac{y^2}{18}=1$ so that the area of the triangle formed by the tangent at P and the co-ordinate axes is the smallest are given by

(a) (2,3) (b) $\sqrt{8},0$ (d) $\left(\sqrt{18},0\right)$ (d) none of these Ans: (a)

6) The difference between the greatest and the least values of the function

$f(x) = \cos x + \frac{1}{2}\cos 2x - \frac{1}{3}\cos 3x$ is

(a) $2/3$ (b) $8/7$ (c) $9/4$ (d) $3/8$ Ans: (c)

7) If $y = a \log|x| + bx^2+x$ has its extreme values at $x = -1$ and $x = 2$ then

(a) a=2 b= -1 (b) a=2, b = $-½$ (c) a=-2, b=½ (d) none of these Ans: (b)

8) If θ is the semivertical angle of a cone of maximum volume and given slant height, then $\tan\theta$ is given by

(a) 2 (b) 1 (c) $\sqrt{2}$ (d) $\sqrt{3}$ Ans: (c)

9) If $f(x) = \frac{x}{\sin x}$ and $g(x) = \frac{x}{\tan x}$ where $0 \le x \le 1$ then in this interval

(a) both f(x) and g(x) are increasing
(b) both f(x) and g(x) are decreasing
(c) f(x) is an increasing function
(d) g(x) is an increasing function Ans: (c)

10) If $f(x) = \begin{cases} 3x^2+12x-1 & : -1 \le x \le 2 \\ 37-x & : 2 < x \le 3 \end{cases}$ then

(a) f(x) is increasing on $[-1,2]$
(b) f(x) is continuous on $[-1,3]$
(c) $f'(2)$ doesn't exist
(d) f(x) has the maximum value at x = 2 Ans: (a),(b),(c),(d)

11) The function $\frac{\sin(x+\alpha)}{\sin(x+\beta)}$ has no maximum or minimum value if

(a) $\beta-\alpha = k\Pi$ (b) $\beta - \alpha \neq k\Pi$ (c) $\beta - \alpha = 2k\Pi$
(d) None of these where K is an integer. Ans: (b)

12) If $f(x)=\dfrac{x^2-1}{x^2+1}$ for every real number then minimum value of f

(a) does not exist (b) is not attained even though f is bounded
(c) is equal to 1 (d) is equal to −1 Ans: (d)

13) If the line ax+by+c = 0 is normal to the curve xy = 1 then

(a) a>0, b>0 (b) a>0, b<0 (d) a<0, b>0 (d) a<0, b<0 Ans: (b),(c)

14) The tangent to the curve

$x=a\sqrt{\cos 2\theta}\cos\theta$ $y=a\sqrt{\cos 2\theta}\sin\theta$ at the point corresponding to $\theta=\Pi/6$ is

(a) Parallel to the x-axis (b) Parallel to the y-axis
(c) Parallel to the line y = x (d) none of these Ans: (a)

15) The minimum value of $f(x) = |3-x| + |2+x| + |5-x|$ is

(a) 0 (b) 7 (c) 8 (d) 10 Ans: (b)

16) If $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$ then $\dfrac{d^2y}{dx^2}$ at $\theta=\Pi/2$ is

(a) $\dfrac{1}{a}$ (b) $\dfrac{1}{2}$ (c) $-\dfrac{1}{a}$ (d) $-\dfrac{1}{2a}$ Ans: (c)

17) If $y=\log\tan\left(\dfrac{\Pi}{4}+\dfrac{\Pi}{2}\right)$ then $\dfrac{dy}{dx}$ is

(a) 0 (b) cosx (c) −secx (d) secx Ans: (d)

18) 'C' on LMV for $f(x) = x^2-3x$ in [0,1] is

(a) 0 (b) $\dfrac{1}{2}$ (c) $-\dfrac{1}{2}$ (d) does not exist Ans: (b)

19) The values of a and b for which the function $f(x)=\begin{cases}ax+1 & x\le 3\\ bx+3 & x>3\end{cases}$ is continuous at x = 3 are

(a) 3a + 2b = 5 (b) 3a = 2+3b (c) 3,2 (d) none of these Ans: (b)

20) The tangents to the curve $y = x^3+6$ at the points (-1,5) and (1,7) are

(a) Perpendicular (b) parallel (c) coincident (d) none of these Ans: (b)

21) If $\dfrac{dy}{dx}=0$ then the tangent is

(a) Parallel to x-axis (b) parallel to y-axis (c) Perpendicular to x-axis
(d) perpendicular to y-axis Ans: (a)

22) If the slope of the tangent is zero at (x_1, y_1) then the equation of the tangent at (x_1,y_1) is

(a) $y_1=mx_1+c$ (b) $y_1= mx_1$ (c) $y-y_1$ (d) y=0 Ans: (c)

23) The function $f(x)=-\dfrac{x}{2}+\sin x$ is always increasing in

(a) $\left(\dfrac{-\Pi}{2},\dfrac{\Pi}{2}\right)$ (b) $\left(0,\dfrac{\Pi}{4}\right)$ (c) $\left(\dfrac{\Pi}{4},\dfrac{\Pi}{2}\right)$ (d) $\left(\dfrac{-\Pi}{3},\dfrac{\Pi}{3}\right)$ Ans: (d)\

24) The least value of 'a' such that the function $f(x) = x^2+ax+1$ is strictly increasing on (1,2) is

(a) −2 (b) 2 (c) $\dfrac{1}{2}$ (d) $-\dfrac{1}{2}$ Ans: (a)

25) The least value of f(x) = tan-1 (sinx + cosx) strictly increasing is

(a) $\left(\frac{\Pi}{4},\frac{\Pi}{2}\right)$ (b) $\left(0,\frac{\Pi}{2}\right)$ (c) 0 (d) none of these

Ans: (d)

4-6 Marks

26) Determine the points of maxima and minima of the function

$f(x)=\frac{1}{8}\log x - bx + cx^2$ where $b \geq 0$

Ans: f has maxima at $2 = \frac{1}{4}\left(b - \sqrt{b^2-1}\right)$ and minima at $\beta = \frac{1}{4}\left(b + \sqrt{b^2-1}\right)$

27) Find the interval in which the following functions are increasing or decreasing

(a) $y = \log\left(x + \sqrt{1+x^2}\right)$ (b) $y = \frac{10}{4x^3 - 9x^2 + 6x}$

Ans: (a) increases on $(-\infty,\infty)$ (b) increases on $\left(\frac{1}{2},1\right)$ and decreases on $(-\infty,0)\cup\left(0,\frac{1}{2}\right)\cup(1,\infty)$

28) Find the equation of normal to the curve $y = (1+x)^y + \sin^{-1}(\sin^2 x)$ at $x = 0$ Ans: x+y=1

29) If P_1 and P_2 are the lengths of the perpendiculars from origin on the tangent and normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ respectively Prove that $4P_1^2 + P_2^2 = a^2$

30) What angle is formed by the y-axis and the tangent to the parabola $y = x^2+4x-17$ at the point $P\left(5/2, -3/4\right)$?

Ans: $\theta = \Pi/2 - \tan^{-1} 9$

31) A cone is circumscribed about a sphere of radius r. Show that the volume of the cone is maximum when its semi vertical angle is $\text{Sin}^{-1}\left(\frac{1}{3}\right)$

32) Find the interval in which the function f(x) is increasing or decreasing

$f(x) = x^3 - 12x^2 + 36x + 17$

Ans:Increasing in x<2 or x>6

Decreasing 2<x<6

33) Find the equation of tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point (x_1, y_1) and show that the sum is intercepts on the axes is constant.

34) Find the point on the curve $y = x^3-11x+5$ at which the tangent has equation $y = x-11$

Ans: (2,−9)

35) Find the equation of all lines having slope −1 and that are tangents to the curve

$y = \frac{1}{x-1}, x \neq 1$

Ans: x+y+1=0 ; x+y−3=0

36) Prove that the curves $y^2 = 4ax$ and $xy = c^2$ cut at right angles if $c^4 = 32a^4$

37) Find the points on the curve $y = x^3 - 3x^2+2x$ at which the tangent to the curve is parallel to the line $y - 2x + 3 = 0$.

Ans: (0,0) (2,0)

38) Show that the semi vertical angle of a right circular cone of maximum volume and given slant height is $\tan^{-1}\sqrt{2}$

39) Show that the semi vertical angle of a right circular cone of given total surface area and maximum volume is $\sin^{-1}\frac{1}{3}$

40) Show that the right circular cone of least curved surface area and given volume is $\sqrt{2}$ times the radius of the base.

41) Show that the height of a cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$

42) Find the absolute maximum and absolute minimum value of $f(x) = 2\cos x + x \quad x \varepsilon [0, \Pi]$

Ans: max at $x = \Pi/6$ and min at $x = 5\Pi/6$

43) Show that the volume of greatest cylinder which can be inscribed in a cone of height h and semi vertical angle α is $\frac{4}{27}\Pi h^3 \tan^2\alpha$

44) A window is in the form of a rectangle above which there is a semicircle. If the perimeter of the window is 'P' cm. Show that the window will allow the maximum possible light only when the radius of the semi circle is $\frac{P}{\Pi + 4}$ cm

45) Find the area of the greatest isoscles triangle that can be inscribed in a given ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its vertex coinciding with one extremity of the major axis.

46) A rectangular window is surmounted by an equilateral triangle. Given that the perimeter is 16 cm. Find the width of the window so that the maximum amount of light may enter.

47) Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semi vertical angle 30° is $\frac{4}{81}\Pi h^3$

48) Show that the height of the right circular cylinder of maximum volume that can be inscribed in a given right circular cone of height h is $h/3$.

49) Of all the rectangles each of which has perimeter 40 metres find one which has maximum area. Find the area also.

50) Show that the rectangle of maximum area that can be inscribed in a circle of radius 'r' cms is a square of side $\sqrt{2}\,r$

51) Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

52) Show that the semi vertical angle of the cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$

53) Given the sum of the perimeters of a square and a circle show that the sum of their areas is least when the side of the square is equal to the diameter of a circle.

54) Find the maximum slope of the curve $f(x) = 2x + 3x^2 - x^3 - 27$

55) Show that the height of the cone of maximum volume that can be inscribed in a sphere of radius 12 cm is 16 cm.

56) A point on the hypotenuse of a right angled triangle is at distances a and b from the sides. Show that the length of the hypotenuse is at least $\left(a^{2/3} + b^{2/3}\right)^{3/2}$

57) Sand is pouring from a pipe at the rate of 12cm³/sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone in increasing when the height is 4 cm?

Ans : $\frac{1}{48\Pi}$ cm/sec.

58) A man 160 cm tall walks away from a source of light situated at the top of the pole 6m high at the rate of 1.1 m/sec. How fast is the length of the shadow increasing when he is 1m away from the pole. Ans: 0.4 cm/sec.

59) An edge of a variable cube is increasing at the rate of 5 cm/sec. How fast is the volume of the cube is increasing when edge is 10cm long? Ans: 1500 cm^3/sec.

60) A balloon which always remains spherical is being inflated by pumping in gas at the rate of 900 cm^3/sec. Find the rate at which the radius of the balloon is increasing when the radius of the balloon is 15 cm. Ans: $\frac{1}{\Pi}$ cm/sec.

61) The volume of a spherical balloon is increasing at the rate of 25 cm^3/sec. Find the rate of change of its surface area at the instant when the radius is 5 cm. Ans: 10 cm^2/sec.

62) The surface area of a spherical bubble is increasing at the rate of 2 cm^2/sec. Find the rate at which the volume of the bubble is increasing at the instant if its radius is 6 cm.

Ans: $80\Pi cm^2$ / sec.

63) Gas is escaping from a spherical balloon at the rate of 900 cm^3/sec. How fast is the surface area, radius of the balloon shrinking when the radius of the balloon is 30cm?

Ans: $\frac{dA}{dt} = 60cm^2$ / sec. $\frac{dr}{dt} = \frac{1}{4\Pi}$ cm/sec.

64) Water is passed into an inverted cone of base radius 5 cm and depth 10 cm at the rate of $\frac{3}{2}$ c.c./sec. Find the rate at which the level of water is rising when depth is 4 cm.

Ans: $\frac{3}{8\Pi}$ cm/sec.

65) Show that the function $f(x) = e^{2x}$ is strictly increasing on R.

66) Show that $f(x) = 3x^5 + 40x^3 + 240x$ is always increasing on R.

67) Find the interval in which the function $f(x) = x^4 - 4x^3 + 4x^2 + 15$ is increasing or decreasing.

68) Find whether $f(x) = \cos\left(2x + \Pi/4\right) \frac{3\Pi}{8} < x < \frac{5\Pi}{8}$ is increasing or decreasing.

69) Find the interval in which the function $\frac{4.\sin x - 2x - x\cos x}{2 + \cos x}$ is increasing or decreasing.

Ans: $\left(0, \Pi/2\right) \cup \left(\frac{3\Pi}{2}, 2\Pi\right)$

70) FInd the interval in which $f(x) = 8 + 36x + 3x^2 - 2x^3$ is increasing or decreasing.

Ans: Increasing : $2 < x < 3$ decreasing : $x > 3$ or $x < -2$

71) Find the interval on which the function $\frac{x}{\log x}$ is increasing or decreasing.

Ans: Increasing in $((e, \infty)$

decreasing for (0,1)U(1,e)

72) Find the points on the curve $\frac{x^2}{4}+\frac{y^2}{25}=1$ at which the tangents are parallel to x-axis and parallel to y-axis. Ans: $(0,\pm5), (\pm2,0)$

73) Find equation of tangents to the curve $x = a\cos^3\theta,\ y = b\sin^3\theta$ at $\theta = \Pi/4$

74) Find the equations of the normal lines to the curve $y = 4x^3-3x+5$ which are parallel to the line $9y+x+3 = 0$. Ans: $x+9y-55=0,\ x+9y-35=0$

75) Find the equation of tangent and normal to the curve $y^2 = \frac{x^3}{4-x}$ at $(2, -2)$

Ans: $2x+y-2=0,\ x-2y-6=0$

76) Find the equation of the tangent to the curve $\sqrt{x}+\sqrt{y} = a$ at the point $\left(\frac{a^2}{4},\frac{a^2}{4}\right)$

Ans: $2x+2y=a^2$

77) Find the angle between the parabolas $y^2=4ax$ and $x^2=4by$ at their point of intersection other than the origin. Ans: $\theta = \tan^{-1}\left[\frac{3a^{1/3}b^{1/3}}{2\left(a^{2/3}+b^{2/3}\right)}\right]$

78) Using differentials find the appropriate value of $(82)^{1/4}$ Ans: 3.0092

79) If $y = x^4-10$ and if x changes from 2 to 1.97, what is the appropriate change in y ?

Ans: -0.96, y changes from 6 to 5.04

80) Find the appropriate change in volume of a cube when side increases by 1%. Ans: 3%

81) Use differentials to evaluate $\left(\frac{17}{81}\right)^{1/4}$ approximately. Ans: 0.677

82) Using differentials evaluate tan 44° approximately, 1° = 0.07145°. Ans: 0.9651

83) Find the approximate value of x if $2x^4-160 = 0$ Ans: 2.991

84) Find the maximum and minimum values of 'f', if any of the function $f(x) = |x|,\ x \varepsilon R$

85) Find the maximum and minimum value of $f(x) = |(\sin 4x+5)|$ without using derivatives.

86) The curve $y = ax^3+6x^2+bx+5$ touches the x-axis at P(-2,0) and cuts the y-axis at a point Q where its gradient is 3. Find a,b,c.

Ans: $a = -\frac{1}{2}\ b = \frac{-3}{4},\ c = 3$

87) Find the local maxima and local minima if any of the function $f(x) = e^{5x}$

88) Find the maxima or minima if any of the function $f(x) = \frac{1}{x^2+2}$

Ans: local max at x = 0, value $\frac{1}{2}$

89) Without using derivatives find the maximum or minimum value of $f(x) = -|x+5| + 3$

Ans: max value 3, no minimum value

RAPID FIRE ON APPLICATIONS OF DERIVATIVES

Q. 1. Find maximum and minimum values of the function $f(x) = 3x^2 + 6x + 8,\ x \in R.$

Q. 2. Prove that the function $f(x) = x^2 - x + 1.$ is neither increasing nor decreasing on (-11).

Q. 3. Find the slope of the tangent to the curve $x^2 + 3y + y^2 = 5\ at\ (1,1).$

Q. 4. The distance moved by a particle traveling in a straight line in t seconds is given by $s = 45t + 11t^2 - t^3$. Find the time taken by the particle to come to the rest.

Q. 5. The radius of a circle is increasing at the rate of 0.7 cm/sec. What is the rate of increase of its circumference?

Q. 6. The radius of a circle is increasing at the rate of 0.1 cm/sec. Determine the rate of change of area when radius of the circle is 5cm.

Q. 7. Find the rate of change of volume of a sphere w.r.t. its surface area, when the radius is 2cm.

Q. 8. Show that the function f (x) = 2x + 3 is strictly increasing on R .

Q. 9. Find the equation of the tangent to the curve $y = -5x^2 + 6x + 7\ at\ the\ point\ \left(\frac{1}{2}, \frac{35}{4}\right).$

Q. 10. Find the interval on which $f(x) = 10 - 6x - 2x^2$ is strictly increasing or decreasing.

Q. 11. For the function $y = x^3 + 21$, find the value of x when y increases 75 times as fast as x .

Q. 12. It is given that at x -1, the function $f(x) = x^4 - 62x^2 + ax + 9$ attains its maximum value on the interval $[0, 2]$. Write the value of a .

Answers

1. Max value 5, Min value does not exist

3. $-\frac{2}{5}$

4. 9 sec

5. 1.4 cm/sec.

6. $\pi\ cm^2$ / sec

7. 1 cm

9. $y = x + \frac{33}{4}$

10. $x > -\frac{3}{2}$ and $x < -\frac{3}{2}$

11. $x = \pm 5$

12. a = 120